

SECOND INFINITESIMAL NEIGHBORHOODS OF PROJECTIVE BUNDLE SECTIONS

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In this paper we will show that given a smooth variety, B , there is a correspondence between projective bundles over B with a section and second infinitesimal neighborhoods of B of the proper dimension. (The terminology and background of this paper will be as is found in [1].) In particular, we will show that a projective bundle over B with a section can be uniquely recovered from the embedding of B in its second infinitesimal neighborhood as a section of the bundle. This follows easily from the following main theorem.

Theorem 1 (Main). *Let B be a smooth variety and let the surjection of locally free sheaves, $\mathcal{E} \twoheadrightarrow \mathcal{O}_B$, determine a projective bundle section, $\sigma : B \rightarrow \Lambda = \mathbb{P}(\mathcal{E})$, where \mathcal{E} is a rank $d + 1$ locally free sheaf. Let \mathcal{I}_B be the ideal sheaf of $\sigma(B)$ in Λ . Then the following short exact sequences of locally free sheaves on B are isomorphic,*

$$\begin{aligned} 0 \rightarrow \mathcal{K} \rightarrow \mathcal{E} \rightarrow \mathcal{O}_B \rightarrow 0 \\ 0 \rightarrow \mathcal{I}_B / \mathcal{I}_B^2 \rightarrow \mathcal{O}_\Lambda / \mathcal{I}_B^2 \rightarrow \mathcal{O}_B \rightarrow 0 \end{aligned}$$

We will start by describing the situation in which we are interested. Let B be a smooth variety. Let \mathcal{E} be a rank $d + 1$ locally free sheaf on B with corresponding projective bundle, $\pi : \Lambda = \mathbb{P}(\mathcal{E}) \rightarrow B$. A section of this bundle, $\sigma : B \rightarrow \Lambda$ is determined by a surjection, $\mathcal{E} \twoheadrightarrow \mathcal{L}$, of \mathcal{E} onto an invertible sheaf \mathcal{L} . By twisting by \mathcal{L}^{-1} , we may assume that $\mathcal{L} = \mathcal{O}_B$ (since $\mathbb{P}(\mathcal{E} \otimes \mathcal{L}^{-1}) \simeq \mathbb{P}(\mathcal{E})$). Let \mathcal{K} be the kernel of the above surjection of sheaves. Then the section $\sigma : B \rightarrow \Lambda$ will correspond to the following short exact sequence of locally free sheaves on B .

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{E} \rightarrow \mathcal{O}_B \rightarrow 0$$

Let $\mathcal{I}_B \subseteq \mathcal{O}_\Lambda$ be the ideal sheaf of $\sigma(B) \subseteq \Lambda$ and let $\mathcal{O}_\Lambda(1)$ be the usual relatively very ample invertible sheaf on Λ . We start with the following proposition.

Proposition 2.

$$\pi_*(\mathcal{I}_B^2(1)) = R^1\pi_*(\mathcal{I}_B^2(1)) = 0$$

Proof. We will work locally on B . Let $\text{Spec } A \simeq U \subseteq B$ be an open affine neighborhood on which \mathcal{E} is free. Then we will have the following correspondences.

- (1) $\mathcal{E}|_U$ with a free module, $A[x_0, \dots, x_d]$,
- (2) $\pi^{-1}(U) \subseteq \Lambda$ with $\text{Proj } A[x_0, \dots, x_d] = \mathbb{P}_A^d$,
- (3) $\mathcal{E}|_U \twoheadrightarrow \mathcal{O}_U$ with a surjection, $A[x_0, \dots, x_d] \twoheadrightarrow A$, $x_i \mapsto s_i$,
- (4) $\mathcal{I}_B|_{\pi^{-1}(U)}$ with $I_B = (s_j x_i - s_i x_j) \subseteq A[x_0, \dots, x_d]$.

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We first wish to describe the sheaf $\mathcal{O}_{\mathbb{P}_A^d}/\mathcal{I}_B^2(1)$ on the variety $\sigma(B)$. Since $A[x_0, \dots, x_d] \rightarrow A$, $x_i \rightarrow s_i$ is a surjection, we can find $\{a_i\} \subseteq A$ such that $\sum a_i s_i = 1$. Define,

$$f := \sum a_i x_i \in A[x_0, \dots, x_d]$$

Then $\sigma(B) \subseteq D_+(f) \simeq \text{Spec } A[x_0, \dots, x_d]_{(f)}$. Thus we can work in this open neighborhood when considering $\mathcal{O}_{\mathbb{P}_A^d}/\mathcal{I}_B^2(1)$. In this neighborhood, \mathcal{I}_B corresponds to,

$$(I_B)_{(f)} = (s_j \frac{x_i}{f} - s_i \frac{x_j}{f}) \subseteq A[x_0, \dots, x_d]_{(f)}$$

Since $\sum a_i s_i = \sum a_i \frac{x_i}{f} = 1$, we observe that for a fixed i ,

$$\sum_j a_j (s_j \frac{x_i}{f} - s_i \frac{x_j}{f}) = \frac{x_i}{f} - s_i$$

Therefore $(I_B)_{(f)}$ can be rewritten as,

$$(I_B)_{(f)} = (y_0, \dots, y_d)$$

where $y_i = \frac{x_i}{f} - s_i$, with the relation $\sum a_i y_i = 0$.

It is clear that $A[x_0, \dots, x_d]_{(f)} = A[y_0, \dots, y_d]$. Therefore $\mathcal{O}_{\mathbb{P}_A^d}/\mathcal{I}_B^2$ corresponds to the module,

$$A[x_0, \dots, x_d]_{(f)}/(I_B)_{(f)}^2 = A[y_0, \dots, y_d]/(y_0, \dots, y_d)^2$$

and $\mathcal{O}_{\mathbb{P}_A^d}/\mathcal{I}_B^2(1)$ corresponds to the module,

$$\overline{f}(A[y_0, \dots, y_d]/(y_0, \dots, y_d)^2)$$

Consider the following long exact sequence of cohomologies on \mathbb{P}_A^d ,

$$0 \rightarrow H^0(\mathcal{I}_B^2(1)) \rightarrow H^0(\mathcal{O}(1)) \rightarrow H^0(\mathcal{O}/\mathcal{I}_B^2(1)) \rightarrow H^1(\mathcal{I}_B^2(1)) \rightarrow H^1(\mathcal{O}(1)) = 0$$

We know that $H^0(\mathcal{O}_{\mathbb{P}_A^d}(1)) = A[x_0, \dots, x_d]_1 = f A[y_0, \dots, y_d]_{\leq 1}$. Therefore the map,

$$H^0(\mathcal{O}_{\mathbb{P}_A^d}(1)) \rightarrow H^0(\mathcal{O}_{\mathbb{P}_A^d}/\mathcal{I}_B^2(1))$$

is an isomorphism of A -modules.

From the above exact sequence it then follows that $H^0(\mathcal{I}_B^2(1)) = 0$ (which we already knew) and that $H^1(\mathcal{I}_B^2(1)) = 0$. Since this is true locally over $U \subseteq B$, the global implication for $\pi : \Lambda \rightarrow B$ is that,

$$\pi_*(\mathcal{I}_B^2(1)) = R^1 \pi_*(\mathcal{I}_B^2(1)) = 0$$

□

(The following theorem is then the same as Theorem 1.)

Theorem 3. *The following short exact sequences of locally free sheaves on B are isomorphic,*

$$\begin{aligned} 0 \rightarrow \mathcal{K} \rightarrow \mathcal{E} \rightarrow \mathcal{O}_B \rightarrow 0 \\ 0 \rightarrow \mathcal{I}_B/\mathcal{I}_B^2 \rightarrow \mathcal{O}_\Lambda/\mathcal{I}_B^2 \rightarrow \mathcal{O}_B \rightarrow 0 \end{aligned}$$

Proof. It is clear after a little thought that the short exact sequence,

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{E} \rightarrow \mathcal{O}_B \rightarrow 0$$

is obtained by applying π_* to the natural sequence,

$$0 \rightarrow \mathcal{I}_B(1) \rightarrow \mathcal{O}_\Lambda(1) \rightarrow \mathcal{O}_B \rightarrow 0$$

(Notice that since σ is determined by $\mathcal{E} \rightarrow \mathcal{O}_B$, we have $\sigma^* \mathcal{O}_\Lambda(1) = \mathcal{O}_B$.)

Now we consider the diagram,

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \rightarrow & \mathcal{I}_B^2(1) & \rightarrow & \mathcal{I}_B^2(1) & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \rightarrow & \mathcal{I}_B(1) & \rightarrow & \mathcal{O}_\Lambda(1) & \rightarrow & \mathcal{O}_B \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \rightarrow & \mathcal{I}_B/\mathcal{I}_B^2 & \rightarrow & \mathcal{O}_\Lambda/\mathcal{I}_B^2 & \rightarrow & \mathcal{O}_B \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

Since the first two rows are exact and the columns are exact, the third row will also be exact. Then applying π_* to the diagram and taking into account the proposition yields the statement in the theorem. \square

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REFERENCES

- [1] Robin Hartshorne. *Algebraic Geometry*. Springer-Verlag, 1977.

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